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The GEMS approach to stationary motions in the spherically symmetric spacetimes

Hong-Zhi Chen and Xing-Chang Song

School of Physics, Peking University
Beijing 100871, P. R. China
E-mail: hzhchen@pku.edu.cn, songxc@pku.edu.cn

Yu Tian and Yi-Hong Gao

Institute of Theoretical Physics, Chinese Academy of Sciences
P.O. Box 2735, Beijing 100080, P. R. China
E-mail: ytian@itp.ac.cn, gaoyh@itp.ac.cn

Abstract: We generalize the work of Deser and Levin on the unified description of Hawking radiation and Unruh effect to general stationary motions in spherically symmetric black holes. We have also matched the chemical potential term of the thermal spectrum of the two sides for uncharged black holes.

Keywords: Models of Quantum Gravity, Black Holes.

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1. Introduction

It is well known that Hawking radiation [1] and Unruh effect [2] are closely related phenomena. Recently, a unified description of the two effects began to emerge. Given that any D-dimensional geometry has a higher dimensional global embedding Minkowski spacetime (GEMS), Deser and Levin [3, 4, 5], found that the temperature detected by the detectors outside the black hole is the same as the temperature detected by the embedded observers in the corresponding GEMS. In their paper, they considered the static observers in various spherically symmetric spacetimes, and the corresponding observers in the GEMS are the usual Rindler observers, i.e. the observers with constant linear acceleration.[1] This correspondence is interesting and profound, and it is worthwhile to extend the discussion to more general detectors, not just the static ones. In this paper, we will generalize this correspondence to general stationary motions in the static spherically symmetric black holes and show that, not only the temperature of the two sides are the same, but also are the whole thermal spectra, including the chemical potential for uncharged black holes.

The outline of the paper is as follows. First, we give the derivation for the Schwarzschild black hole. Then we generalize it to the more complicated RN-AdS black hole. And finally we discuss some related issues.

2. The Schwarzschild black hole

We only consider the four dimensional cases. We first consider the simplest Schwarzschild black hole, and then generalize it to more general spherically symmetric case. Generally, a static spherically symmetric black hole has four Killing vectors, one is timelike corresponding to the time-translational invariance and the other three are spacelike corresponding to rotational invariance. In the previous works, they only considered detectors following

[1] The GEMS approach has also been detailedly studied by many other authors. See [6, 7, 8], for example.

the integral curves of the timelike Killing vector field ∂_t, i.e. static detectors. In this case, the same detector in the GEMS will be in a motion of constant linear acceleration, i.e. a standard Rindler detector. We want to consider a more general case in which the motion of the detectors are stationary in the sense that they follow the integral curves of a general timelike Killing vector field, as thus the corresponding motion in the GEMS is also a stationary one, which we can deal with. In the case of the Schwarzschild black hole, it is a circular motion with constant r and θ in the usual coordinates.

To verify the correspondence, we have to compute the thermal spectra from the two sides and see if they are the same. We begin with the black hole side.

2.1 The black hole side

The Schwarzschild metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.1}$$

We consider the observer following the trajectory

$$r = r_0 > 2M\,, \qquad \theta = \theta_0\,, \qquad \phi = \Omega t\,, \tag{2.2}$$

where r_0, θ_0 and Ω are constants.[2] The authors of [5] only considered the case $\Omega = 0$. To determine the spectrum observed by such an observer, we can not use the usual methods directly because the observer is moving while the usual methods are applied to a static observer. However, we can make the following coordinate transformation to the observer's rest frame,

$$\tilde{\phi} = \phi - \Omega t\,. \tag{2.3}$$

In this new frame, the metric becomes (taking $(x^0, x^1, x^2, x^3) = (t, r, \theta, \tilde{\phi})$)

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r} - r^2\Omega^2 \sin^2\theta\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \\ &\quad - r^2(d\theta^2 + \sin^2\theta d\tilde{\phi}^2) - 2r^2\Omega \sin^2\theta d\tilde{\phi} dt\,. \end{aligned} \tag{2.4}$$

Note that this metric is stationary and axisymmetric. The detector with trajectory (2.2) is now at the position of

$$r = r_0 > 2M\,, \qquad \theta = \theta_0\,, \qquad \tilde{\phi} = 0\,. \tag{2.5}$$

There are different methods to calculate the Hawking temperature of a rest detector in this case. We will apply the method of Damour and Ruffini [9].[3] First, from the condition

$$\hat{g}_{00} \equiv g_{00} - \frac{g_{03}^2}{g_{33}} = 0\,, \tag{2.6}$$

[2] We take $\Omega > 0$, without loss of generality. And, of course, we should take $\Omega r_0 \sin\theta_0 < 1$ to avoid unphysical motions.

[3] See also [10] for extensive discussions of D-R method.

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it is easy to see that the horizon of metric (2.4) is at

$$r_{\mathrm{h}} = 2M\,. \tag{2.7}$$

And from the following formula

$$\kappa = \lim_{\hat{g}_{00}\to 0}\left(-\frac{1}{2}\sqrt{\frac{g^{11}}{-\hat{g}_{00}}}\hat{g}_{00,1}\right), \tag{2.8}$$

we get the surface gravity of the horizon:

$$\kappa = \frac{1}{4M}\,. \tag{2.9}$$

Both of these two quantities are the same as those of the usual Schwarzschild black hole, as expected.

We then consider the massless Klein-Gordon equation in this spacetime, which is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\nu}}\right) = 0\,. \tag{2.10}$$

We separate the variables as follows,

$$\phi(t,r,\theta,\tilde{\phi}) = e^{-\mathrm{i}(\omega t - m\tilde{\phi})}\chi(\theta)\psi(r)\,, \tag{2.11}$$

where ω and m is the coordinate energy and the magnetic quantum number of the particles, respectively. We further define the Regge-Wheeler radial coordinate,

$$r_{*} = r + 2M\ln\frac{r-2M}{2M}\,, \tag{2.12}$$

and put eqs. (2.11), (2.12) into eq. (2.10). After separating the variables, we finally obtain the radial equation

$$\frac{d^{2}\psi(r_{*})}{dr_{*}^{2}} + \frac{4(r-2M)}{r^{2}}\frac{d\psi(r_{*})}{dr_{*}} + \left[(\omega+m\Omega)^{2} + \frac{2M-r}{r^{3}}\lambda\right]\psi(r_{*}) = 0\,, \tag{2.13}$$

where λ is the constant from separating variables. When taking the limit $r \to r_{\mathrm{h}}$, this equation becomes

$$\frac{d^{2}\psi(r_{*})}{dr_{*}^{2}} + (\omega+m\Omega)^{2}\psi(r_{*}) = 0\,. \tag{2.14}$$

Following [9], we deduce that the detector will detect a thermal spectrum of the form

$$N_{\omega} = \frac{1}{e^{(\omega+m\Omega)/T_{0}} - 1}\,, \tag{2.15}$$

where

$$T_{0} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}\,, \tag{2.16}$$

and N_{ω} is the mean number of particles with energy ω. However, we note that the temperature T_{0}, energy ω and angular velocity Ω are all coordinate quantities. To get the corresponding local quantities detected by the detector at position (2.5), we just divide them by the red shift factor from eq. (2.4), and finally get

$$N_{\tilde{\omega}} = \frac{1}{e^{(\tilde{\omega}+m\tilde{\Omega})/T} - 1}\,, \tag{2.17}$$

where now

$$T = \frac{T_0}{f} = \frac{1}{8\pi M f}\,, \qquad \tilde{\omega} = \frac{\omega}{f}\,, \qquad \tilde{\Omega} = \frac{\Omega}{f}\,, \tag{2.18}$$

and

$$f(r_0, \theta_0, \Omega) = \sqrt{g_{00}} = \sqrt{1 - \frac{2M}{r_0} - r_0^2\Omega^2 \sin^2\theta_0}\,. \tag{2.19}$$

Of course, the parameters r_0, θ_0 and Ω are arbitrary, and this formula applies to any detector with trajectory (2.2).

2.2 The GEMS side

The above Schwarzschild space can be embedded in a flat $D = 6$ Minkowski spacetime with metric

$$ds^2 = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 - (dz^5)^2\,, \tag{2.20}$$

as follows [11]:

$$\begin{aligned} z^0 &= 4M\sqrt{1 - \frac{2M}{r}} \sinh\left(\frac{t}{4M}\right), \\ z^1 &= 4M\sqrt{1 - \frac{2M}{r}} \cosh\left(\frac{t}{4M}\right), \\ z^2 &= \int dr \sqrt{\frac{2Mr^2 + 4M^2 r + 8M^3}{r^3}}\,, \\ z^3 &= r\sin\theta\sin\phi\,, \\ z^4 &= r\sin\theta\cos\phi\,, \\ z^5 &= r\cos\theta\,. \end{aligned} \tag{2.21}$$

It should be emphasized that this embedding can be extended to cover $r < 2M$.[4] We see that a detector following the trajectory (2.2) will effectively be an ordinary 4-dimensional Rindler motion superposed with a circular motion whose plane is perpendicular to the acceleration of the Rindler motion. Because when $\Omega = 0$, the results of [5] indicate that the Unruh detector detects the same temperature as that of the Hawking detector in the Schwarzschild space, intuitively, we expect this detector will detect the same spectrum as eq. (2.17) when $\Omega \neq 0$, since both sides we are considering now are superposed with the same circular motion. Although this is not a standard Rindler motion, we can still apply the method of Bogoliubov transformation in [2] to calculate the spectrum detected by such a detector. The quantization of field theory in general stationary coordinate systems

[4]The issue of the complete global embedding had been discussed in [12].

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had been discussed previously in [13, 14]. Their discussions have been extended in [15], including detailed discussions on the causal structures. To apply the results of [15], we have to determine what their parameters a and ω correspond to in our case.

First, we briefly summarize their results relevant to us. It is easy to see that our case corresponds to $a \neq 0$, $\omega_x \neq 0$ and $\omega_z = 0$ in their paper, where a is the proper acceleration of the detector along x-direction and ω_x is the proper angular velocity of the detector in the plane perpendicular to $x-$ axis. For such a motion, the trajectory is easily found to be

$$\begin{aligned} t' &= C \sinh(a\tau)\,, \\ x' &= C \cosh(a\tau) + D\,, \\ r' &= r_0\,, \\ \phi' &= \omega_x \tau\,, \end{aligned} \tag{2.22}$$

where t', x', r' and ϕ' are the cylindrical coordinates in the 4-dimensional Minkowski space, C, D and r_0 are constants, and τ is the proper time of the detector. In this case, there are horizons in the detector's rest frame and the vacuum state is defined by the standard canonical quantization procedure in this frame. Unlike in the standard Rindler motion, the restriction of the positive norm modes to one side of the horizon does not at the same time restrict the positive frequency modes to the same side. So the vacuum thus defined will not be the true ground state and will have negative energy excitations. They found that the detector will detect a thermal spectrum in the Minkowski vacuum state and the spectrum is of the form

$$N_\varepsilon = \frac{1}{e^{(\varepsilon + m\omega_x)/T_\mathrm{e}} - 1}\,, \tag{2.23}$$

where

$$T_\mathrm{e} = \frac{a}{2\pi}\,, \tag{2.24}$$

m is the magnetic quantum number of the particle, and ε is the local energy.

Now, we come to our embedding space. For a detector following the trajectory (2.2), the metric (2.20) becomes

$$ds^2 = \left(1 - \frac{2M}{r_0} - r_0^2 \Omega^2 \sin^2\theta_0\right) dt^2\,, \tag{2.25}$$

so we see that t is not the proper time. However, it is easily converted to the proper time by a rescaling

$$t = \frac{\tilde{t}}{f}\,, \tag{2.26}$$

where f is defined in eq. (2.19). After such a rescaling, the trajectory of the embedded detector becomes[5]

$$z^0 = 4M\sqrt{1 - \frac{2M}{r_0}} \sinh\left(\frac{\tilde{t}}{4Mf}\right)\,,$$

[5]Here we have combined z^3 and z^4 into polar coordinates R and Φ.

$$
\begin{aligned}
z^1 &= 4M\sqrt{1-\frac{2M}{r_0}}\cosh\left(\frac{\tilde{t}}{4Mf}\right), \\
R &= \text{constant}, \\
\Phi &= \Omega'\tilde{t}, \\
z^2 &= \text{constant}, \qquad z^5 = \text{constant},
\end{aligned} \tag{2.27}
$$

where

$$
\Omega' = \frac{\Omega}{f}. \tag{2.28}
$$

Comparing eq. (2.18) and eq. (2.28), we see that

$$
\tilde{\Omega} = \Omega'. \tag{2.29}
$$

From eq. (2.22) and eq. (2.27), we see that in our case

$$
a = \frac{1}{4Mf}, \qquad \omega_x = \Omega' = \tilde{\Omega}. \tag{2.30}
$$

So from eq. (2.23), this detector will detect the thermal spectrum

$$
N_\varepsilon = \frac{1}{e^{(\varepsilon+m\Omega')/T_{\mathrm{e}}}-1} \tag{2.31}
$$

with

$$
T_{\mathrm{e}} = \frac{a}{2\pi} = \frac{1}{8\pi Mf}. \tag{2.32}
$$

Comparing these results with those of Schwarzschild space eqs. (2.17), (2.18), we find that they are identical. Thus, we have verified that the correspondence is valid in this more general case, including the chemical potential term.

3. The RN-AdS case

We will now consider the more general spherically symmetric space, namely RN-AdS black hole.[6] The metric of 4-dimensional RN-AdS space is [16]

$$
ds^2 = F(r,M,Q,R)dt^2 - F^{-1}(r,M,Q,R)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}
$$

where

$$
F(r,M,Q,R) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}. \tag{3.2}
$$

The outer horizon $r = r_{\mathrm{h}}$ of this space is determined from

$$
F(r_{\mathrm{h}},M,Q,R) = 0, \tag{3.3}
$$

and the corresponding surface gravity κ is

$$
\kappa = \frac{r_{\mathrm{h}}^2 - Q^2 + 3r_{\mathrm{h}}^4 R^{-2}}{2r_{\mathrm{h}}^3}. \tag{3.4}
$$

[6] We restrict our discussion to the non-extremal case.

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The authors of [17] had discussed the GEMS of RN-AdS space. They found that it can be embedded in a flat $D = 7$ space with metric

$$ds^2 = (dz^0)^2 - \sum_{i=1}^{5}(dz^i)^2 + (dz^6)^2\,, \tag{3.5}$$

as follows:

$$\begin{aligned}
z^0 &= \kappa^{-1}\sqrt{F(r,M,Q,R)}\,\sinh(\kappa t)\,,\\
z^1 &= \kappa^{-1}\sqrt{F(r,M,Q,R)}\,\cosh(\kappa t)\,,\\
z^2 &= z^2(r,r_{\rm h},Q,R)\,,\\
z^3 &= r\sin\theta\cos\phi\,,\\
z^4 &= r\sin\theta\sin\phi\,,\\
z^5 &= r\cos\theta\,,\\
z^6 &= z^6(r,r_{\rm h},Q,R)\,,
\end{aligned} \tag{3.6}$$

where z^2 and z^6 are complicated functions of the arguments displayed whose explicit forms do not concern us.[7] Note that this embedding is very similar to that of the Schwarzschild space (2.21), except for the extra time dimension z^6 which is unimportant for the case we will consider. Consider the following trajectory of a detector:

$$r = r_0 > r_{\rm h}\,, \qquad \theta = \theta_0\,, \qquad \phi = \Omega t\,, \tag{3.7}$$

where r_0, θ_0 and Ω are constants. We see from eq. (3.6) that the embedded motion again reduces effectively to a 4-dimensional one with z^2, z^5, z^6 being constants. From the similarity of the metric and embedding of the RN-AdS space with those of the Schwarzschild space, we can easily show that the local temperature T calculated form the two sides are the same, and both are

$$T = \frac{\kappa}{2\pi f}\,, \tag{3.8}$$

where

$$f(r_0,\theta_0,\Omega) = \left(1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} + \frac{r_0^2}{R^2} - r_0^2\Omega^2\sin^2\theta_0\right)^{\frac{1}{2}}. \tag{3.9}$$

While for the chemical potential term, the issue is more subtle. Because the RN black hole is charged, it will induce a chemical potential term in the thermal spectrum even for a static detector. However, on the GEMS side as discussed already by [5], it is a standard Rindler motion and the detector will not detect a chemical potential in the thermal spectrum. Thus even in that case, the chemical potential is not properly matched. We found that this is always the case for charged black hole because the embedding does not map all the information of the black hole into the higher dimensional spacetime, namely the electric field of the black hole does not map to anything in the higher dimensional space. Obviously, to match the chemical potential of the two sides in the case of charged black hole properly, we have to settle this problem first.

[7] Again, see [12] for extending the embedding to $r < r_{\rm h}$ case.

4. Concluding remarks

We have generalized the unified description of Hawking radiation and Unruh effect proposed by Deser and Levin to the general stationary motions. For the case of uncharged black hole, we also matched the chemical potential of the two sides. Our discussion is also easily extended to pure dS/AdS space. The issue of matching the chemical potential for the charged black hole needs further investigations to find a way of properly mapping all the information contained in the charged case to the higher dimensional embedding space. This unified picture of the two effects seems to be two points of view of the same thing. The detector is the same one, just the points of view are different. In one point of view it moves in a curved spacetime and will detect Hawking radiation associated with the horizon due to gravity, while in the other one it moves in a higher dimensional Minkowski space (possibly with more times) and will detect radiation associated with the horizon due to acceleration. From the previous calculations, we see that the key of the unification is the correspondence of the quantum states of the two sides. For example, in the Schwarzschild case, the detector is taken to be in the Hartle-Hawking vacuum; while in the GEMS side, the detector is taken to be in the Minkowski vacuum. It is this mapping of the two states that makes the two points of view gives the same results. Our generalization implies that this unified description is not just a coincidence and there may be something important in it. The correspondence between a lower dimensional quantum effect and a higher dimensional quantum effect may also have something to do with the holographic principle. The deep reason of this correspondence deserves further study.

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